

Requirements for Defining a Function

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To define a function $g : A \rightarrow B$, one must:

- 1) Specify the Domain set A precisely.
- 2) Specify the Co-domain set B precisely.
- 3) Specify (by formula or by describing a process) the method for determining $g(z)$, $\forall z \in A$ (the Domain set)

Note: Parts 1) and 2) are accomplished by using the phrase “Define function $g : A \rightarrow B$ ” as long as it is clear what particular sets A and B are.

Example 1: The following correctly defines a function named “ T ”:

“The function $T: \mathbb{Z} \rightarrow \mathbb{R}$ is defined as follows:

For each $x \in \mathbb{Z}$, define $T(x) = x^2 + 1$.”

You read the above definition using these words:

“Define the function T from the set of integers to the set of real numbers as follows:

For each x , x an element of \mathbb{Z} , define ‘ T of x ’ to be equal to $x^2 + 1$.”

This definition specifies that the domain of T is \mathbb{Z} and that the codomain of T is \mathbb{R} . It does this by the notation “ $T: \mathbb{Z} \rightarrow \mathbb{R}$.” It then specifies, for each element in the domain of T (here, \mathbb{Z}), how to determine which element in the codomain of T (here, \mathbb{R}) is to be related by T to that element in the domain of T .

Example 2: Sometimes there are varying procedures for determining the related value $f(x)$ for a given value of x , depending on which subset it is, in a partition of the domain f , that that particular value of x lies.

For example, function $P: \mathbb{Z} \rightarrow \mathbb{Z}$ defined below is a function such that all positive even numbers are related to 10, all non-positive even numbers are related to 5, and all odd numbers are related to 0.

“Define the function $P: \mathbb{Z} \rightarrow \mathbb{Z}$ as follows: For all $x \in \mathbb{Z}$, define $P(x) = 10$ if x is even and positive; define $P(x) = 5$ if x is even and not positive; and define $P(x) = 0$ if x is odd.”

This type of “piecewise defined” function definition is usually presented using the following table format:

$$P(x) = \begin{cases} 10, & \text{if } x \text{ is even and } x > 0. \\ 5, & \text{if } x \text{ is even and } x \leq 0. \\ 0, & \text{if } x \text{ is odd.} \end{cases}$$

Some Common Errors Made When Trying to Define Functions:

1. Failing to specify the domain and codomain. For example,

“Define f to be the function, $f(x) = \frac{x^2-1}{x}$.”

(In some textbooks, there is a convention which says that, when a function is defined in this truncated way, the domain is considered to be the set of all real numbers such that the formula defining $f(x)$ computes a real number. There is no convention as to what the codomain is beyond that of any set containing the actual range of the function.)

2. Failing to show how to compute $f(x)$ for every element of the domain:

For example, “Define $f: \mathbb{Z} \rightarrow \mathbb{Z}$ as follows: For all $x \in \mathbb{Z}$, $f(x) = \begin{cases} x^2, & \text{if } x > 0. \\ 1-x, & \text{if } x < 0. \end{cases}$

(Here, $x = 0$ has been left out and so $f(0)$ has not been defined, which is an error.

3. Making the calculation of $f(x)$ ambiguously indeterminate:

For example: Define $f: \mathbb{Z}^+ \rightarrow \mathbb{R}$ as follows:

For all $n \in \mathbb{Z}$, define $f(n) = x$ where $x^2 = n$.

You can see the difficulty when you try to figure out what the value of $f(n)$ ought to be when $n = 4$. When $n = 4$, there are two possible values, $x = 2$ and $x = -2$, such that $x^2 = 4$. So, it may be that $f(4) = 2$ or it could be that $f(4) = -2$; we just don't know which.

Finally, the phrase “**Evaluate the function f at $x = 2$** ” is an instruction to determine what particular value y in the range is the image of 2 under f . We evaluate the function f at “ a ” when we calculate what $f(a)$ is. A common sloppy paraphrase of this is:

“Plug 2 into the function and see what you get.”